Combinatorial Hopf algebras in particle physics II

Erik Panzer

Scribed by Iain Crump

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1 Toy Model of Divergences

We will take rooted trees (instead of Feynman graphs) to explore the nesting of divergences. The Feynman rules are $\phi_S : \mathcal{H}_R \to \mathcal{A}$.

Example.

$$\phi_S(\bullet) = \int_0^\infty \frac{dx}{x + S} \text{ (divergent)}$$

$$\phi_{R,S}(\bullet) = \phi_S(\bullet) - \phi_\mu(\bullet) = \int_0^\infty \left[\frac{dx}{x + S} - \frac{dx}{x + \mu} \right] = -\log \frac{S}{\mu} = -l,$$

where $l = log \frac{S}{\mu}$. We call $\phi_{R,S}$ the renormalized Feynman rules.

Example.

$$\phi_S \left(\begin{smallmatrix} & \bullet x & \\ y \bullet & & \bullet z \end{smallmatrix} \right) = \int_0^\infty \frac{dx}{x + S} \int_0^\infty \frac{dy}{y + x} \int_0^\infty \frac{dz}{z + x} \text{ (really divergent)}$$

$$\phi_{R,S} \left(\begin{smallmatrix} & \bullet x & \\ y \bullet & & \bullet z \end{smallmatrix} \right)$$

$$= \int_0^\infty \left[\frac{dx}{x + S} - \frac{dx}{x + \mu} \right] \int_0^\infty \left[\frac{dy}{y + x} - \frac{dy}{y + \mu} \right] \int_0^\infty \left[\frac{dz}{z + x} - \frac{dz}{z + \mu} \right]$$

$$= -\frac{1}{3} l^3 - \frac{\pi^2}{3} l$$

Definition. Let f be piecewise continuous on $[0, \infty)$ with $f(\zeta) \in \mathcal{O}\left(\frac{1}{\zeta}\right)$, then define the Feynman rules to be the *character* $(\zeta \to \infty)$ such that

$$\phi_S(B_+(f)) := \int_0^\infty \frac{f(\zeta/S)}{S} d\zeta \; \phi_\zeta(f).$$

A character $\phi : \mathcal{H}_R \to \mathcal{A}$ is a multiplicative map; ie. $\phi(ab) = \phi(a)\phi(b)$.

Lemma 1 (Universal property of $\mathcal{H}_R$)**.** *For any commutative algebra $\mathcal{A}$ and linear map $L : \mathcal{A} \to \mathcal{A}$ there is exactly one character $L_\rho : \mathcal{H}_R \to \mathcal{A}$ such that $L_\rho \circ B_+ = L \circ L_\rho$.*

Definition. The renormalization character $\phi_{R,S} : \mathcal{H}_R \to \mathcal{A}$ is determined by

$$\phi_{R,S}(B_+(f)) = \int_0^\infty \left[\frac{f(\zeta/S)}{S} - \frac{f(\zeta/\mu)}{\mu} \right] \phi_{R,S}(f) \, d\zeta.$$

Lemma 2. *If $f(\zeta)_{(\zeta \to \infty)} = c_{-1}\zeta^{-1} + \mathcal{O}(\zeta^{-1-\epsilon})$ for $\epsilon > 0$, then $\phi_{R,S}(f)$ are convergent integrals and evaluate to polynomials in $l = \log\frac{S}{\mu}$.*

Proof. We do induction, and need only consider trees. So, assume $t = B_+(w)$ such that $\phi_{R,S}(\widetilde{w}) \in \mathbb{C}[l]$ for all forests $\widetilde{w}$ with $|\widetilde{w}| \leq |w|$. Then,

$$\phi_{R,S}(t) = \int \left(\frac{f(\zeta/S)}{S} - \frac{f(\zeta/\mu)}{\mu} \right) d\zeta \; \underbrace{\phi_{R,S}(w)}_{\sum a_i l_i}$$

$$= \int_0^\infty \left[\frac{f(\zeta/(S/\mu))}{S/\mu} - f(\zeta) \right] \underbrace{\log^n \zeta}_{\substack{(d_{-z})^n|_{z=0}\zeta^{-z} \\ = e^{-z\log\zeta}}} d\zeta$$

$$= (d_{-z})^n \, |_{z=0} \left\{ \underbrace{\int_0^\infty \frac{f(\zeta/(S/\mu))}{S/\mu} \zeta^{-z} \, d\zeta}_{(S/\mu)^{-z} F(z)} - \underbrace{\int_0^\infty f(\zeta)\zeta^{-z} \, d\zeta}_{F(z)} \right\}$$

$$= (d_{-z})^n \, |_{z=0} \left\{ F(z)(e^{-zl} - 1) \right\}.$$

$\square$

We call $F(z)$ the Mellin transform;

$$F(z) = \int_0^\infty f(\zeta)\zeta^{-z}\,d\zeta = \sum_{n\geq 1} c_n z^n$$

$$= (-1)^n n! \sum_{i\geq 1} \frac{(-l)^i}{i!} c_{n-i}$$

$$= \sum_{i\geq 1} (-d_l)^{n-i} c_{n-i}(l^n)$$

$$= P \circ F(-d_l)(l^n)$$

where $P = \mathrm{id} - e = \mathrm{id} - \mathbf{1}\epsilon$.

Corollary 3. $\phi_{R,S}(B_+(w)) = (P \circ F(-d_l))[\phi_{R,S}(w)]$.

Example.

$$\phi_{R,S}(\bullet) = \underbrace{P \circ F(-d_l)}_{c_{-1}(-d_l)^{-1}+c_0+c_1(-d_l)+\cdots} \qquad \underbrace{\phi_{R,S}(\mathbf{1})}_{1} = -\int_o^l c_{-1}\cdot 1 dl^1 = -c_{-1}l$$

$$\phi_{R,S}\left(\bigwedge\right) = P \circ F(-d_l)\underbrace{[\phi_{R,S}(\bullet)]^2}_{c_{-1}^2 l^2} = c_{-1}^3\frac{1}{3}l^3 + c_0 c_{-1}^2 l^2 - 2lc_1 c_{-1}^2$$

1.1 Birkhoff decomposition

When we subtract divergences we need to give them physical meaning or sense. If the method of renormalization is ad hoc, then, it will create problems in trying to make sense of the operation physically. The goal in renormalizable field theory is that all these things that you absorb you can explain by redefining a constant in your field theory. Hence, we must understand the structure of the things we subtract. As such, we introduce Brikhoff decomposition.

Consider decomposed into two subalgebras $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$ and a character $\phi : \mathcal{H}_R \to \mathcal{A}$. Then, there exists a unique pair $\phi_+, \phi_- : \mathcal{H}_R \to \mathcal{A}$ of characters such that

1. $\phi = \phi_-^{*-1} * \phi_+$

2. $\phi_\pm(\ker \epsilon) \subseteq \mathcal{A}_\pm$

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Recall that $\mathcal{H}_R = \mathbb{Q}\mathbf{1} + \underbrace{\bigoplus_{n>0} \mathcal{H}_{R,n}}_{\xrightarrow{\epsilon} 0}$.

In $\mathcal{H}_R$, our goal should be a subalgebra of convergent integrals. So we start with some character which is not of a nice type, and we have to specify what we are allowed to subtract or what we are allowing for the infinities. Here $\mathcal{A}_-$ would be the counter-terms, or the infinite ones. Once you define how you split the algebra into good things and bad things, there is a unique way to rewrite your original character as a product of one which only takes values of the good subalgebra and another which is only takes values from this subalgebra of counter-terms.

We learn that it is always possible to decompose a character like this. Hence, everything infinite that we subtract can be factored off from the renormalized Feynman rules via multiplication. This is the key ingredient for the physical interpretation. We will see this more tomorrow when we consider Dyson Schwinger equations. This is the precisely the property that allows you to "eat up" all divergences in a redefinition of the physical parameter.

There is a general method as to how you can find ϕ_- and ϕ_+, but we will focus on the things we have constructed already. We have a renormalized Feynman rule and an original Feynman rule, and we know about the counter-terms; the graphs where we put $S = \mu$. So we will show that it holds for our construction.

Lemma 4. $\phi_S = \phi_\mu * \phi_{R,S}$. *Equivalently,* $\phi_{R,S} = \phi_\mu^{*-1} * \phi_S$

Proof. We will use induction similar to what we have seen before. As we are dealing with characters, everything is multiplicative, we need only prove this for trees;

$$(\phi_\mu^{*-1} * \phi_S)B_+ = m(\phi_\mu^{*-1} \otimes \phi_S)\underbrace{\Delta B_+}_{(\mathrm{id}\otimes B_+)\Delta + B_+\otimes\mathbf{1}}$$

$$\phi_\mu^{*-1} * (\phi_S \circ B_+) + \underbrace{\phi_\mu^{*-1} \circ B_+}_{\underbrace{\phi_\mu \circ S \circ B_+}_{-S*B_+}} = \phi_\mu^{*-1} * (\phi_S \circ B_+ - \phi_\mu \circ B_+)$$

$$= \int \left[\frac{f(\zeta/S)}{S} - \frac{f(\zeta/\mu)}{\mu} \right] d\zeta \underbrace{\phi_\mu^{*-1} * \phi_\zeta}_{\phi_{R,S}}$$

$$= \phi_{R,S} \circ B_+.$$

$\square$

Remark 5. This is very useful, even in general situations. For any algebra take this splitting however you want, and you have a factorization property. That is, there are other methods of renormalization that what we did; subtract by setting S to a particular value μ, subtracted at a particular reference point (renormalization by subtraction). For example, you can try to write everything in a regulator like we did intermediately where you have an additional parameter Z. Then you have poles, and you can think of subtracting these poles and again using this construction.

2 Regularization

This is of combinatorial interest. We have said that we may treat divergent integrals as formal objects. We want to give this quantitative meaning, though; how divergent it is, which of two is more divergent. So we introduce a new parameter which makes everything is well-defined but you can see how divergent you are. There are different methods to do this, we will consider only one.

Definition. Let $_z\phi_S$ be the character such that

$$_z\phi_S \circ B_+ = \int_0^\infty \frac{f(\zeta/S)}{S} d\zeta\, _z\phi_S \cdot \zeta^{-z}.$$

Naturally, if we take $z = 0$ we get what we had before. We will want to take the limit as $z \to 0$, if $z > 0$ this makes the integral convergent. We can then see what happens as z approaches zero.

Lemma 6. *For forest w, $_z\phi_S(w) = S^{-z|w|} \prod_{v \in V(w)} F(z| \underbrace{w_v}_{\text{subtree that hangs below } v} |).$*

Example.

$$_z\phi_S(\bullet) = S^{-z}F(z)$$

$$_z\phi_S\left(\bigwedge\right) = S^{-3z}F(3z)F(z)F(z)$$

With this, we can very easily compute the renormalized Feynman rules. As $_z\phi_{R,S} = \,_z\phi_\mu^{*-1} * \,_z\phi_S,$

$$_z\phi_{R,S}\left(\vline\right) = \,_z\phi_\mu\left(-\vline + \bullet\bullet\right) + \,_z\phi_S\left(\vline\right) - \,_z\phi_\mu(\bullet)\,_z\phi_\mu(\bullet)$$

$$= (S^{-2z} - \mu^{-2z})F(2z)F(z) + F(z)F(z)(\mu^{-2z} - \mu^{-z}S^{-z}).$$

We can do this for any tree, but the functions F start with a pole; $F(2z) = \frac{c_{-1}}{2z} + c_0 + c_1(2z) + \cdots$. Looking at individual terms, there are poles in z as they are Laurent series in z. But we have proven that if we forget about z then everything is finite and we get well-defined answers. We know that $_z\phi_{R,S}$ is free of poles because $\lim_{z\to 0} {}_z\phi_{R,S} = \phi_{R,S}$. Hence, all the poles cancel.

This relates to combinatorial identities (see exercises).

3   Renormalization Group

The following material is needed for the exercises. It will be covered in greater detail in the next lecture.

Lemma 7 (Universal property of $\mathcal{H}_R$)**.** *If $L : \mathcal{A} \to \mathcal{A}$ is a cocycle (ie. $\Delta L = (id\otimes L)\Delta + L\otimes\mathbf{1}$), then there exists a distinct character $L_\rho : \mathcal{H}_R \to \mathcal{A}$, $L_\rho \circ B_+ = L \circ L_\rho$ with the property $\Delta L_\rho = (L_\rho \otimes L_\rho)\Delta$.*

We have a character $\phi_{R,S} : \mathcal{H}_R \to \mathbb{Q}[l]$. Note that $\mathbb{Q}[l]$ is a Hopf algebra $(\Delta l = l \otimes \mathbf{1} + \mathbf{1} \otimes l)$.

Proposition 8. $\Delta\phi_{R,S} = (\phi_{R,S} \otimes \phi_{R,S})\Delta$

Hint: we know $\phi_{R,S} = L_\rho$, where $L = P \circ F(-d_l)$.

Example.

$$\phi_{R,l}\left(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\right) = c_{-1}^2\frac{l^2}{2} - c_{-1}c_0 l$$

$$\phi_{R,a} * \phi_{R,b}\left(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\right) = c_{-1}^2\frac{a^2 + b^2}{2} - c_{-1}c_0(a + b) + c_{-1}^2 ab$$

$$= \phi_{R,a+b}\left(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\right)$$

An implication is that the full Feynman rules are completely determined by the linear bits.

$$\left.\begin{array}{c}\phi_{R,a} = ev_a \circ \phi_R \\ \phi_{R,b} = ev_b \circ \phi_R\end{array}\right\} \quad \phi_{R,a} * \phi_{R,b} = m(ev_a \circ \phi_R \otimes ev_b \circ \phi_R)\Delta$$

$$= \underbrace{(ev_a * ev_b)}_{ev_{a+b}} \circ \phi_R$$

Definition. Let G denote the space of characters $\mathcal{H}_R \to \mathbb{Q}[x]$, and g denote the infinitesimal characters; $g = \{f \mid f : \mathcal{H}_R \to \mathbb{Q}[x], f(ab) = f(a)\epsilon(b) + f(b)\epsilon(a)\}$. Then there are bijections

$$\exp_* := \left(f \mapsto \sum_{n \geq 0} \frac{f^{*n}}{n!} \right) : g \to G,$$

$$\log_* := \left(f \mapsto -\sum_{n \geq 1} \frac{(f - e)^{*n}}{n} \right) : G \to g.$$

Lemma 9. $\phi_{R,S} = \exp_*(-\gamma l)$ (ie. $\gamma : \mathcal{H}_R \to \mathbb{C}$, $\gamma := -d_l \mid_{l=0} \phi_{R,S}$)

Hence, γ just takes the linear piece, and the full Feynman rules are the exponential.

Example.

$$\phi_{R,S}\left(\begin{smallmatrix} \bullet \\ | \\ \bullet \end{smallmatrix} \right) = c_{-1}^2 \frac{l^2}{2} + l...$$

$$= \exp_*(-\gamma l) = ...l^2 \frac{\gamma * \gamma \left(\begin{smallmatrix} \bullet \\ | \\ \bullet \end{smallmatrix} \right)}{2}...$$

We know that these can be derived from the linear term in the Feynman rules of the single vertex tree.